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Optimal Discrete-Time Static Output-Feedback Design: A w-Domain Approach

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Presented in this paper is an alternative design methodology for optimal discrete-time control-law synthesis (i..e, what we call a W-design method). The proposed method solves an optimal static output-feedback design for sampled-data systems with a given known sampling time. The overall scheme involves the formulation of an exact optimal control problem for discrete-time systems using a state-space transformation and an equivalent quadratic cost function in the w domain. A one-to-one relationship between a set of stabilizing feedback gains in the w domain and those in the traditional z domain has been established. The optimal output-feedback gain obtained from the minimization of the equivalent cost functional expressed in the w domain will yield the same optimum solution as the one obtained traditionally in the z domain. An attractive feature of this procedure is that design solutions can be obtained using only continuous-time-domain techniques. The procedure is demonstrated with the synthesis of a longitudinal-axis stability augmentation system for a commercial transport.

Introduction

TRANSFORM methods have been widely used and discussed in textbooks on the subject of sampled-data control systems.[1-3] However, usage of w transform, as shown in the literature,[1,4] has been mostly limited to classical control-law synthesis based on Bode frequency responses and root locus,[5] and in spectral factorization algorithms.[4] Direct use of the w transform has not been fully explored to our knowledge in the solution of multivariable optimal discrete-time problems. Recently, H^∞ designs in sampled-data control systems have been obtained in Ref. 6 using the w transform in combination with recent results on H^∞ control for continuous-time systems. Thus, it is perceived that advances in optimal control techniques for problems defined in the continuous-time domain can be readily extended to discrete-time problems using the w transform, in particular, the synthesis of robust low-order controllers using parameter optimization.[7] What is lacking in the literature is the exact formulation of such a problem and a procedure to solve for the optimum solution. The additional feature of the W-design method addressed here is the fact that this method would allow us to treat discrete-time control-law synthesis with frequency-dependent design specifications. Furthermore, frequency loop-shaping techniques for sampled-data systems may be developed based on results of loop transfer recovery developed in the continuous-time domain.[8,9]

Moreover, the proposed algorithm seems to offer a noticeable improvement in the numerical accuracy for discrete-time problems even for the case of small sampling time. The problem for an optimal static output-feedback design in a multivariable sampled-data system can be expressed entirely in terms of state-space equations, thereby resulting in a design procedure that is efficient and the solution amenable to numerical computations. The purpose of this paper is to provide the exact formulation of a linear quadratic optimal discrete-time problem in the w domain, and its solution using a numerical optimization technique developed for continuous-time problems.[7]

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Characteristics of the w Transform

Common treatment of a sampled-data system in the frequency domain involves the introduction of the z transform. Other well-known frequency-domain mappings have also been discussed in textbooks.[1,2] Early definition of such a mapping from the z domain to the w domain has the form of a bilinear transformation defined by $(z-1)/(z+1)$ where $z=e^{sT}$. This mapping, however, is not appropriate since the transformed system does not reduce to the continuous-time system as the sampling time T approaches zero. A slightly modified but convenient transform, again called w transform in Ref. 1, is defined by

$$w = \frac{2}{T} \times \frac{z-1}{z+1}$$

which was also used in Ref. 5. Some useful properties of this w transform are summarized next.

Property 1: When the sampling time T approaches zero, the w operator becomes the Laplace operator s; that is, the transfer function in the w domain, $G_w(w)$, actually becomes the continuous-time transfer function, $G(s)$, as the sampling time T goes to zero.

Property 2: Let $w=\eta+j\nu$ and $s=\sigma+j\omega$ then $\nu=2/T\tan(\omega T/2)$.

Property 3: The inverse of w (i.e., w^{-1}) is closely analogous to s^{-1} for a simple integrator if the sampling time T is small.

Property 4:

$$z^{-1} = \frac{2/T - w}{2/T + w}$$

is analogous to a first-order Pade approximation of the time delay

$$e^{-sT} \approx \frac{2/T - s}{2/T + s}$$

Property 5: The frequency ν introduces the warping effect on the continuous-time frequency ω due to the sampling scheme. For $|\omega| < \pi/4T$, then $\nu \cong \omega$; for example, if $\omega = \pi/4T$, then

$$\frac{\nu}{\omega} = \frac{\tan(\pi/8)}{\pi/8} = 1.0548 \ (\cong 5\% \text{ warping})$$

Property 6: System stability in the w domain is defined by having eigenvalues located in the left half of the w plane where the real part η of the eigenvalue $w=\eta+j\nu$ is negative. Stability region inside the unit circle of the z plane is now mapped into the left half of the w plane using the w transform. This is similar to the stability region defined for continuous-time systems.

State-Space Model of a Plant in the w Domain

Let a continuous-time plant state model be given by

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \tag{1}$$

with initial conditions $x(0)$ where $x(t)$ is a state vector of dimension n, $u(t)$ is an input vector of dimension m, and $y(t)$ is an output vector of dimension p. The state matrices A, B, C, and D are constant matrices in $\mathcal{R}^{n\times n}$, $\mathcal{R}^{n\times m}$, $\mathcal{R}^{p\times n}$, and $\mathcal{R}^{p\times m}$, respectively. Assuming that the plant inputs and outputs are sampled at a constant sampling interval T and the inputs $u(t)$ to the continuous-time plant are held constant over the sampling interval (i.e., using a zero-order hold device), the discrete-time state equations of the resulting sampled-data system are

$$x_{k+1} = A_d x_k + B_d u_k$$
$$y_k = C_d x_k + D_d u_k \tag{2}$$

where

$$A_d = e^{AT}, \quad B_d = \int_0^T e^{At}\,\mathrm{d}t B, \quad C_d = C, \quad D_d = D \tag{3}$$

Note that the discrete-time transfer function is given by $G_z(z) = C_d(zI_n - A_d)^{-1}B_d + D_d$ where $z = e^{sT}$. For a discrete-time system given in Eq. (2), and applying the w transform defined in the previous section, we obtain the following equivalent set of state equations in the w domain. By introducing a "fictitious" state vector $x_w(w)$ of dimension n, we have

$$wx_w(w) = A_w x_w(w) + B_w u(w) + \Gamma_w x_o \tag{4}$$

$$y(w) = C_w x_w(w) + D_w u(w) + G_w x_o \tag{5}$$

And the original state vector $x(w)$ is now an output of the preceding system model with the following equation,

$$x(w) = x_w(w) + E_w u(w) + \Omega_w x_o \tag{6}$$

where

$$A_w = (2/T)(A_d + I_n)^{-1}(A_d - I_n)$$
$$B_w = (4/T)(A_d + I_n)^{-2}B_d$$
$$\Gamma_w = (4/T)(A_d + I_n)^{-2}A_d$$
$$C_w = C_d$$
$$G_w = C_d(A_d + I_n)^{-1}$$
$$D_w = D_d - C_d(A_d + I_n)^{-1}B_d$$
$$E_w = -(A_d + I_n)^{-1}B_d$$
$$\Omega_w = (A_d + I_n)^{-1} \tag{7}$$

And conversely, we have

$$A_d = \left[(2/T)I_n + A_w\right]\left[(2/T)I_n - A_w\right]^{-1}$$
$$B_d = (4/T)\left[(2/T)I_n - A_w\right]^{-2}B_w$$
$$C_d = C_w$$
$$D_d = D_w + C_w\left[(2/T)I_n - A_w\right]^{-1}B_w \tag{8}$$

Furthermore, for convenience, we define a new output $y_w(w)$ to be

$$y_w(w) \equiv y(w) - D_w u(w) = C_w x_w(w) + G_w x_o \tag{9}$$

where I_n is an identity matrix of dimension n. Here, $x(w)$, $u(w)$, and $y(w)$ are, respectively, the w transformed variables of the states x_k, inputs u_k, and outputs y_k. The defined outputs $y_w(w)$ are similar to the outputs $y(w)$, except that they do not contain the direct feedthrough term from the inputs $u(w)$. The significance of these newly defined outputs will be evident in the theorem where we establish a one-to-one equivalence between a feedback design in the w domain using outputs $y_w(w)$ and a feedback design in the z domain using outputs $y(z)$.

It should be noted that the discrete-time plant model in Eq. (2) is represented exactly in the w domain by Eqs. (4) and (5) for any given sampling time T. As pointed out before, the transformed variable for the state vector x_k is given in Eq. (6); it can be treated as an output vector $y(w)$ with $C_w = I_n$, $D_w = E_w$, and $G_w = \Omega_w$. For strictly proper systems (i.e., $D = D_d = 0$), the system equations (4) and (5) will always have at least one nonminimum-phase zero at $w = 2/T$. This fundamental property derives directly from the sampler-and-hold used in sampled-data control systems. Since these devices are inherent in any sampled-data system, performance of the resulting control design will therefore be limited by the presence of these nonminimum phase zeros, in comparison with that achieved under an analog (i.e., continuous-time) design. Following are some remarks on the w-domain model.

Remark 1: Let λ_z and λ_w be eigenvalues of the system matrices A_d and A_w, respectively. It can be shown that

$$\lambda_w = \frac{2}{T}\frac{\lambda_z - 1}{\lambda_z + 1}$$

From this relationship, we can deduce that the real part of the eigenvalue λ_w is negative (i.e., stable), if and only if the magnitude of the eigenvalue $\lambda_z < 1$. This can be easily seen from the fact that

$$\eta = \mathrm{Re}(\lambda_w) = \frac{2}{T}\frac{|\lambda_z|^2 - 1}{|\lambda_z|^2 + 2\mathrm{Re}(\lambda_z) + 1} \leq 0$$

implies $|\lambda_z| \leq 1$, where Re( ) means the real part of ( ).

Remark 2: Even when the plant transfer function matrix between the inputs $u(t)$ (or u_k) and the outputs $y(t)$ (or y_k) [i.e., $G(s)$ or $G_z(z)$] is strictly proper with $D = D_d = 0$, the corresponding transfer function matrix expressed in the w domain will always be proper with a nonzero direct feedthrough matrix D_w. The transfer function matrix in the w domain is defined by

$$G_w(w) = C_w(wI_n - A_w)^{-1}B_w + D_w$$

Remark 3: As the sampling time T approaches zero, it can be shown that the system matrices A_w and B_w will tend to be continuous-time state matrices A and B, respectively, and the direct feedthrough matrix D_w will become D. Note that

$$A_d = e^{AT} = I + AT + \frac{A^2T^2}{2!} + \cdots$$

and

$$B_d = \int_0^T e^{At}\,dt B = BT + \frac{T^2}{2!}AB + \cdots$$

then from Eq. (7) we have

$$A_w = (1/T)(AT + \cdots), \qquad B_w = (1/T)(BT + \cdots)$$

$$D_w = D - \tfrac{1}{2}(BT + \cdots)$$

Remark 4: Frequency responses of the transfer function $G_w(w)$ in the w domain (i.e., $w = j\nu$, $0 \le \nu < \infty$) match closely those of the continuous-time transfer function $G(s)$ (i.e., $s = j\omega$, $0 \le \omega \ll \pi/4T$).

Realization of a w-Domain Output-Feedback Design

Consider a plant state model given by Eqs. (4) and (5); let us assume that one has determined an internally stabilizing output-feedback control-law $u(w) = K_w y_w(w)$ using any available design methods such as output-feedback eigenassignment, parameter optimization,[7] Riccati-based methods, and others. Note that the static output-feedback gain K_w is applied to the output $y_w(w)$ defined in Eq. (9). To physically implement such an output-feedback control law synthesized in the w domain, one needs to convert it back into the z domain and obtain an equivalent gain K_d in the control-law $u(z) = K_d y(z)$. This relation would allow us to implement *any* (not necessarily optimal) output-feedback control-law K_w. The result is given in the following theorem.

Theorem: If an output-feedback control-law K_w in $u(w) = K_w y_w(w)$ stabilizes the system given in Eqs. (4) and (9), then, the following output-feedback control-law

$$K_d = (I_m + K_w D_w)^{-1} K_w$$

in

$$u(z) = K_d y(z)$$

will stabilize the discrete-time system given in Eq. (2) where D_w is defined in Eq. (7). Furthermore, the eigenvalues of the closed-loop systems with feedback laws $u(w) = K_w y_w(w)$ in the w domain and $u(z) = K_d y(z)$ in the z domain are related by the w transform (see Remark 1).

Proof: Assume that the closed-loop system with feedback law $u(w) = K_w y_w(w)$ is stable; i.e., all of the eigenvalues of $(A_w + B_w K_w C_w)$ lie in the left half of the w plane. Then it remains to show that with the feedback $u(z) = K_d y(z)$ where

$$K_d = (I_m + K_w D_w)^{-1} K_w$$

the eigenvalues of the corresponding closed-loop system will also lie inside the unit circle in the z plane. Notice that one can solve for K_w in terms of K_d and obtain

$$K_w = (I_m - K_d D_w)^{-1} K_d$$

Considering the closed-loop system matrix $(A_w + B_w K_w C_w)$, we have

$$\begin{aligned} A_w + B_w K_w C_w &= (2/T)(A_d + I_n)^{-1}(A_d - I_n) \\ &+ (4/T)(A_d + I_n)^{-2} B_d \Big[I_m + K_d \{ C_d (A_d + I_n)^{-1} B_d \\ &- D_d \}^{-1} K_d \Big] C_d \\ &= (2/T)(A_d + I_n)^{-1} \Big[\{ A_d + B_d (I_m - K_d D_d)^{-1} K_d C_d \} \\ &- I_n \Big] \cdot \Big[\{ A_d + B_d (I_m - K_d D_d)^{-1} K_d C_d \} + I_n \Big]^{-1} (A_d + I_n) \\ &= (2/T)(A_d + I_n)^{-1}(A_{dcl} - I_n)(A_{dcl} + I_n)^{-1}(A_d + I_n) \end{aligned} \tag{10}$$

where

$$A_{dcl} = A_d + B_d (I_m - K_d D_d)^{-1} K_d C_d \tag{11}$$

is the closed-loop system matrix of the z-domain model.

Let V be a similarity transformation that places the closed-loop system matrix A_{dcl} into an upper-triangular form. Now, define the matrix $Z = V^{-1} A_{dcl} V$ where Z is an upper-triangular matrix with closed-loop eigenvalues z_i $(i = 1, n)$ along its diagonal, then we have

$$\begin{aligned} A_w + B_w K_w C_w &= (A_d + I_n)^{-1} V \\ &\times \{ (2/T)(Z - I_n)(Z + I_n)^{-1} \} V^{-1} (A_d + I_n) \end{aligned} \tag{12}$$

The eigenvalues of $(A_w + B_w K_w C_w)$ are simply those of $(2/T)(Z - I_n)(Z + I_n)^{-1}$. Therefore, the eigenvalues z_i $(i = 1, n)$ of Z are related to the eigenvalues w_i $(i = 1, n)$ of $(A_w + B_w K_w C_w)$ through the w transform; i.e.,

$$w_i = \frac{2}{T}\frac{z_i - 1}{z_i + 1}$$

Thus, if the real parts of the eigenvalues w_i $(i = 1, n)$ are negative, then, according to Remark 1, the corresponding eigenvalues z_i $(i = 1, n)$ will also lie inside the unit circle and are therefore stable.

We have shown that, if an output-feedback gain matrix K_w stabilizes a system described in the w domain, then the discrete-time feedback gain matrix $K_d = (I_m + K_w D_w)^{-1} K_w$ will stabilize the same system represented in the z domain. This result enables us to develop design methods in the w domain having desired performance and robustness properties, and later implement these designs using the above theorem. For example, a design procedure for a stabilizing output-feedback gain in the w domain can be developed using Riccati-based methods to achieve desirable loop-shaping, and then we apply the theorem to realize such a design in the discrete-time domain.

Quadratic Cost Function in the w Domain

Let us consider a control problem involving the minimization of a discrete-time cost function J_d given by

$$J_d = \frac{1}{2} \sum_{k=0}^{\infty} E\{ x_k^T Q_w x_k + 2 u_k^T M_w x_k + u_k^T R_w u_k \} \tag{13}$$

which is subject to the state constraints given in Eq. (2), where $E\{\ \}$ means the expectation operator. The state initial conditions x_o are treated as random with zero mean $E\{x_o\} = 0$ and covariance $E\{x_o x_o^T\} = X_o$. The matrices A_d, B_d, C_d, and D_d are given in Eq. (3). Note that the state equations in Eq. (2) along with the weighting matrices Q_w, M_w, and R_w must satisfy the usual stabilizability and detectability conditions[10,11] for the existence of an optimal stabilizing state-feedback law. The resulting optimal control law[11] is given by

$$u_k = K_d x_k \tag{14}$$

with

$$K_d = -(R_w + B_d^T S B_d)^{-1}(M_w^T + B_d^T S A_d) \tag{15}$$

and S satisfies the steady-state discrete-time Riccati equation[12]

$$\begin{aligned} S &= A_d^T S A_d + Q_w - (M_w^T + B_d^T S A_d)^T \\ &\times (R_w + B_d^T S B_d)^{-1}(M_w^T + B_d^T S A_d) \end{aligned} \tag{16}$$

To establish an equivalent formulation in the w domain for Eq. (13), we invoke the Parseval theorem and the w transform. The cost function J_d can be rewritten as

$$J_d = \frac{1}{4\pi j}\int_{-j\infty}^{+j\infty} E\left\{\begin{bmatrix}x(w)\\u(w)\end{bmatrix}^*\begin{bmatrix}Q_w & M_w\\M_w^T & R_w\end{bmatrix}\begin{bmatrix}x(w)\\u(w)\end{bmatrix}\right\} \times \frac{4/T}{(2/T)^2-(w)^2}\,\mathrm{d}w \tag{17}$$

where [ ]* implies the conjugate transpose of a vector or a matrix. Letting $w = j\nu$, Eq. (17) takes on the familiar form of a Fourier integral given by

$$J_d = \frac{1}{2\pi}\int_{0}^{\infty} E\left\{\begin{bmatrix}x(j\nu)\\u(j\nu)\end{bmatrix}^*\begin{bmatrix}Q_w & M_w\\M_w^T & R_w\end{bmatrix}\begin{bmatrix}x(j\nu)\\u(j\nu)\end{bmatrix}\right\} \times \frac{4/T}{(2/T)^2+\nu^2}\,\mathrm{d}\nu \tag{18}$$

Minimization of the cost function given in Eq. (17) is subjected to the following set of constraints: 1) state constraints in the w domain given in Eqs. (4) and (9) where x_o represents the set of random plant initial conditions and 2) constraint on the controller structure for a static output-feedback design,

$$u(w) = K_w y_w(w) \tag{19}$$

where K_w is an output-feedback gain matrix determined from the minimization of the cost function J_d in Eq. (17).

It is interesting to note that the optimal feedback gain K_w in Eq. (19), even for the full-state-feedback case (i.e., $C_d = I_n$ and $D_d = 0$), cannot be solved from continuous-time algebraic Riccati equations. Basically, when formulated in the w domain, the problem becomes one that corresponds to an output-feedback design as depicted in Fig. 1. By examining the cost function in Eq. (17), we see that one needs to introduce shaping filters for the system state variables x and the control variables u. The "time constant" of the shaping filter depends only on the sampling time T. This filter is crucial to the exact formulation of the cost function J_d in Eq. (13) for control problems expressed in the w domain. From Eq. (18), one can see clearly the direct dependence of the design performance on the sampling time T. In general, one would resort to numerical search techniques for solving the optimal control gain matrix K_w. Effective numerical algorithms have been developed based on gradient search, which requires the availability of the necessary conditions for optimality. These conditions are developed in the next section.

Necessary Conditions for Optimality

To solve the problem stated in the preceding section, it is convenient first to reformulate the system model into a fictitious time-domain representation with a time variable τ where the operator dτ assumes a similar function as the continuous-time differential operator dt. More precisely, we define the w transform in the τ domain as follows:

$$F(w) = \mathfrak{F}(f(\tau)) \equiv \int_{0^-}^{\infty} f(\tau)e^{-w\tau}\,\mathrm{d}\tau \tag{20}$$

where $\mathfrak{F}(\mathrm{d}f/\mathrm{d}\tau) = wF(w) - f(0^-)$.

Using a time-domain definition with the variable τ above, an augmented state model for the optimal W-design procedure can be obtained as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\chi_w(\tau) = \hat{A}\chi_w(\tau) + \hat{B}u(\tau) + \hat{\Gamma}x_o\delta(\tau) \tag{21}$$

with $\chi_w(0^-) = 0$ where $\chi_w(\tau) = [x_w^T(\tau), x_f^T(\tau), u_f^T(\tau)]^T$ and $\delta(\tau)$ is the well-known Dirac function. The additional states $x_f(\tau)$ and $u_f(\tau)$ are used to implement the shaping filters on the states $x(\tau)$ and inputs $u(\tau)$ for the cost function J_d of Eq. (17). The state matrices $\hat{A}$, $\hat{B}$, and $\hat{\Gamma}$ can be constructed,

$$\hat{A} = \begin{bmatrix} A_w & \mathbf{0} & \mathbf{0} \\ \frac{2}{\sqrt{T}}I_n & -\frac{2}{T}I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{2}{T}I_m \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} B_w \\ \frac{2}{\sqrt{T}}E_w \\ \frac{2}{\sqrt{T}}I_m \end{bmatrix}, \qquad \hat{\Gamma} = \begin{bmatrix} \Gamma_w \\ \frac{2}{\sqrt{T}}\Omega_w \\ \mathbf{0} \end{bmatrix} \tag{22}$$

The output-feedback law given in Eq. (19) becomes

$$u(\tau) = K_w y_w(\tau) \tag{23}$$

where

$$\begin{aligned} y_w(\tau) &= [C_d \quad \mathbf{0} \quad \mathbf{0}]\chi_w(\tau) + G_w x_o\delta(\tau) \\ &= \hat{C}\chi_w(\tau) + G_w x_o\delta(\tau) \end{aligned} \tag{24}$$

The cost function given in Eq. (17) can be expressed as

$$\begin{aligned} J_d(K_w, x_o) &= \frac{1}{2}\int_0^{\infty} E\{\chi_w^T\Sigma_w\chi_w\}\,\mathrm{d}\tau \\ &= \frac{1}{2}\mathrm{Tr}\left(\Sigma_w\int_0^{\infty} E\{\chi_w\chi_w^T\}\,\mathrm{d}\tau\right) \end{aligned} \tag{25}$$

where

$$\Sigma_w = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q_w & M_w \\ \mathbf{0} & M_w^T & R_w \end{bmatrix} \tag{26}$$

The above formulation corresponds exactly to the optimal control problem in the continuous-time domain[13] for an optimal output-feedback design with a quadratic cost function. Thus, one can make use of solution methods that are developed for continuous-time systems to solve the discrete-time problem at hand. Note that, with the augmented state model, the cost function in Eq. (25) contains a penalty on the state variables $\chi_w(\tau)$ and does not contain a direct penalty on the control inputs $u(\tau)$ (i.e., it has the form of a singular cost). Gradients of the cost function $J_d(K_w, x_o)$ with respect to the

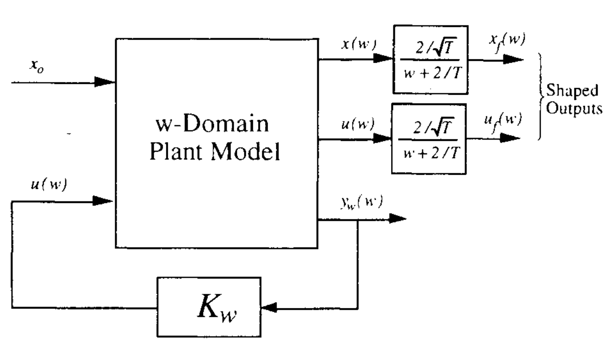

Fig. 1 Schematic block diagram in the W-design methodology.

feedback gain matrix K_w can be derived using, for example, results of Ref. 7. We obtain

$$\frac{\partial J_d}{\partial K_w} = \hat{B}^T \Lambda \{X \hat{C}^T + (\hat{\Gamma} + \hat{B} K_w G_w) X_o G_w^T\} \tag{27}$$

where X and Λ are solutions of the following Lyapunov equations:

$$(\hat{A} + \hat{B} K_w \hat{C})^T \Lambda + \Lambda(\hat{A} + \hat{B} K_w \hat{C}) + \Sigma_w = \mathbf{0}$$

$$(\hat{A} + \hat{B} K_w \hat{C}) X + X(\hat{A} + \hat{B} K_w \hat{C})^T + (\hat{\Gamma} + \hat{B} K_w G_w) X_o (\hat{\Gamma} + \hat{B} K_w G_w)^T = \mathbf{0} \tag{28}$$

The cost function J_d becomes $J_d = ½\mathrm{Tr}\{\Sigma_w X\}$. At the optimum design solution, one must have $\partial J_d / \partial K_w = 0$.

The necessary conditions for optimality are similar in form to those obtained for the continuous-time problems.[13] The significance of the equations given above are that they apply instead to the solutions of an optimal static output-feedback design for discrete-time systems. These equations can, of course, be solved using numerical algorithms developed for continuous-time problems.[7] Some remarks concerning the design procedure based on Eqs. (27) and (28) are given next.

Remark 5: Numerous well-known results can be derived from the stated set of necessary conditions for optimality. For example, when simplified to the state-feedback case, Eq. (27) together with Eq. (28) and the gain relation shown in the theorem will reduce to the familiar discrete-time Riccati equation given in Eqs. (15) and (16). Note that, in full-state feedback case, the optimal gain K_w is also independent of the initial state covariance matrix X_o. However, in general this is not true for the output-feedback case.

Remark 6: From the results obtained so far, we have provided an alternative exact formulation of H^2 optimization problem for sampled-data systems. The problem is defined and solved completely in the w domain.

Remark 7: Results can be easily extended to problems involving discrete-time stochastic systems excited by disturbances ζ_k of zero mean $E\{\zeta_k\} = 0$ and covariance $E\{\zeta_i \zeta_j^T\} = W_o \delta_{ij}$. The discretized state-model at a sampling time T is given by

$$x_{k+1} = A_d x_k + B_d u_k + \Omega_d \zeta_k, \qquad y_k = C_d x_k + D_d u_k \tag{29}$$

and, without loss of generality, we assume $x_o = \mathbf{0}$. A state-space representation of Eq. (29) in the w domain is

$$w x_w(w) = A_w x_w(w) + B_w u(w) + \Gamma_w \zeta(w)$$

$$y_w(w) = C_w x_w(w) + G_w \zeta(w) \tag{30}$$

where A_w, B_w, and C_w are the same as in Eq. (7), while the matrices Γ_w and G_w are given by

$$\Gamma_w = (4/T)(A_d + I_n)^{-2}\Omega_d, \quad G_w = -C_d(A_d + I_n)^{-1}\Omega_d \tag{31}$$

In this case, the cost function J_d is of the form

$$J_d = ½ \lim_{\tau \to \infty} E\{\chi_w^T(\tau)\Sigma_w \chi_w(\tau)\} = ½\,\mathrm{Tr}\left(\Sigma_w \lim_{\tau\to\infty} E\{\chi_w(\tau)\chi_w^T(\tau)\}\right) \tag{32}$$

The necessary conditions for optimality given in Eqs. (27) and (28) also apply if we let $X_o = W_o$ and use Γ_w and G_w as given in Eq. (31). The cost function J_d in Eq. (17) is replaced by Eq. (32).

Remark 8: The preceding results also can be extended to the dynamic output-feedback case for a system in Eqs. (4) and (9) excited by random initial conditions x_o. Let us suppose that the dynamic compensator is of order r and has the following form:

$$w x_\tau(w) = A_\tau x_\tau(w) + B_\tau y_w(w)$$

$$u(w) = C_\tau x_\tau(w) + D_\tau y_w(w) \tag{33}$$

State of the augmented model is simply $\chi_z = [\chi_w^T, x_\tau^T]^T$ and all of the controller state matrices of Eq. (33) are compactly placed into the output-feedback gain matrix K_w as follows[14]:

$$K_w \equiv \begin{bmatrix} D_\tau & C_\tau \\ B_\tau & A_\tau \end{bmatrix} \tag{34}$$

In this case, the cost function J_d is of the form

$$J_d(K_w, x_o) = \frac{1}{2}\int_0^\infty E\{\chi_z^T \Sigma_z \chi_z\}\,\mathrm{d}\tau = \frac{1}{2}\mathrm{Tr}\left(\Sigma_z \int_0^\infty E\{\chi_z \chi_z^T\}\,\mathrm{d}\tau\right) \tag{35}$$

where

$$\Sigma_z \equiv \begin{bmatrix} \Sigma_w & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r\times r} \end{bmatrix} \tag{36}$$

The necessary conditions for optimality given in Eqs. (27) and (28) also apply to the case of a dynamic compensator if we make the following substitutions,

$$\hat{A} \equiv \begin{bmatrix} \hat{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r\times r} \end{bmatrix}, \quad \hat{B} \equiv \begin{bmatrix} \hat{B} & \mathbf{0} \\ \mathbf{0} & I_{r\times r} \end{bmatrix}, \quad \hat{C} \equiv \begin{bmatrix} \hat{C} & \mathbf{0} \\ \mathbf{0} & I_{r\times r} \end{bmatrix}$$

$$\hat{\Gamma} \equiv \begin{bmatrix} \hat{\Gamma} \\ \mathbf{0}_{r\times n} \end{bmatrix}, \quad G_w \equiv \begin{bmatrix} G_w \\ \mathbf{0}_{r\times n} \end{bmatrix}, \quad \Sigma_w \equiv \begin{bmatrix} \Sigma_w & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r\times r} \end{bmatrix} \tag{37}$$

Numerical Example

In this section, we synthesize a stability augmentation system for a commercial transport B767 using the W-design method. Different output-feedback controller structures have been investigated. A longitudinal aircraft dynamic model is obtained for the flight condition: weight 184,000 lb, Mach 0.80, altitude 35,000 ft, velocity 778.3 ft/s, and c.g. location 0.18 MAC (mean aerodynamic chord). It has the form of Eq. (1) with states $x(t) = \{V(\mathrm{ft/s}), \alpha(\mathrm{deg}), q(\mathrm{deg/s}), \theta(\mathrm{deg})\}$, input $u(t) = \{\delta_e(\mathrm{deg})\}$ and outputs $y(t) = \{V(\mathrm{ft/s}), \alpha(\mathrm{deg}), q(\mathrm{deg/s}), \theta(\mathrm{deg})\}$. The state matrices A, B, C, and D are given by

$$A = \begin{bmatrix} -1.6750\times10^{-2} & 1.1214\times10^{-1} & 2.8000\times10^{-4} & -5.6083\times10^{-1} \\ -1.6400\times10^{-2} & -7.7705\times10^{-1} & 9.9453\times10^{-1} & 1.4700\times10^{-3} \\ -4.1670\times10^{-2} & -3.6595\times10^{+0} & -9.5443\times10^{-1} & 0 \\ 0 & 0 & 1.0000\times10^{+0} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2.4320\times10^{-2} \\ -6.3390\times10^{-2} \\ -3.6942\times10^{+0} \\ 0 \end{bmatrix}, \qquad C = I_4, \qquad D = \mathbf{0}_{4\times1}$$

We consider the following continuous-time cost function J,

$$J = \frac{1}{2}\int_0^\infty E\{x^T(t)Qx(t) + u^T(t)Ru(t)\}\,\mathrm{d}t \tag{38}$$

where $Q = I_4$ and $R = 1$, and the covarince matrix of the initial condition x_o is $X_o = I_4$. The above continuous-time system and its performance index J is then discretized using a sampling time $T = 0.03$ s.

The weighting matrices Q_w, M_w, and R_w in the cost function J_d of Eq. (13) are obtained according to the discretization method described in Ref. 1 taking into account the intersample behavior

$$Q_w = \begin{bmatrix} 2.9985\times10^{-2} & 4.4026\times10^{-5} & -1.9975\times10^{-5} & -2.5243\times10^{-4} \\ 4.4026\times10^{-5} & 2.9396\times10^{-2} & -1.1541\times10^{-3} & -1.6134\times10^{-5} \\ -1.9975\times10^{-5} & -1.1541\times10^{-3} & 2.9143\times10^{-2} & 4.4573\times10^{-4} \\ -2.5243\times10^{-4} & -1.6134\times10^{-5} & 4.4573\times10^{-4} & 3.0003\times10^{-2} \end{bmatrix}$$

$$M_w = \begin{bmatrix} -9.5713\times10^{-6} \\ 7.4031\times10^{-5} \\ -1.6141\times10^{-3} \\ -1.6381\times10^{-5} \end{bmatrix}, \qquad R_w = \begin{bmatrix} 3.0120\times10^{-2} \end{bmatrix}$$

With the given discrete-time state model, we then formulate the synthesis model in the w domain according to Eq. (7):

$$A_w = \begin{bmatrix} -1.6752\times10^{-2} & 1.1201\times10^{-1} & 2.5379\times10^{-4} & -5.6083\times10^{-1} \\ -1.6409\times10^{-2} & -7.7770\times10^{-1} & 9.9463\times10^{-1} & 1.4691\times10^{-3} \\ -4.1671\times10^{-2} & -3.6599\times10^{+0} & -9.5510\times10^{-1} & 3.5299\times10^{-6} \\ -7.5394\times10^{-6} & -4.7512\times10^{-4} & 1.0002\times10^{+0} & -1.3493\times10^{-6} \end{bmatrix}$$

$$B_w = \begin{bmatrix} -2.4332\times10^{-2} \\ -9.5065\times10^{-3} \\ -3.7514\times10^{+0} \\ 5.5152\times10^{-2} \end{bmatrix}$$

$$\Gamma_w = \begin{bmatrix} 3.3333\times10^{+1} & 6.7237\times10^{-4} & 3.3734\times10^{-3} & -7.1701\times10^{-5} \\ 2.1308\times10^{-4} & 3.3356\times10^{+1} & 1.2915\times10^{-2} & -6.0478\times10^{-5} \\ -7.5415\times10^{-4} & -4.7528\times10^{-2} & 3.3354\times10^{+1} & -1.3492\times10^{-4} \\ 3.1253\times10^{-4} & 2.7452\times10^{-2} & 7.1683\times10^{-3} & 3.3333\times10^{+1} \end{bmatrix}$$

$$E_w = \begin{bmatrix} 3.6668\times10^{-4} \\ 9.5822\times10^{-4} \\ 5.5424\times10^{-2} \\ 4.2320\times10^{-6} \end{bmatrix}$$

$$G_w = \begin{bmatrix} 5.0013\times10^{-1} & -8.4009\times10^{-4} & -1.9034\times10^{-6} & 4.2062\times10^{-3} \\ 1.2307\times10^{-4} & 5.0583\times10^{-1} & -7.4597\times10^{-3} & -1.1019\times10^{-5} \\ 3.1253\times10^{-4} & 2.7449\times10^{-2} & 5.0716\times10^{-1} & -2.6474\times10^{-8} \\ 5.6546\times10^{-8} & 3.5634\times10^{-6} & -7.5015\times10^{-3} & 5.0000\times10^{-1} \end{bmatrix}$$

$$D_w = \begin{bmatrix} 3.6668\times10^{-4} \\ 9.5822\times10^{-4} \\ 5.5424\times10^{-2} \\ 4.2320\times10^{-6} \end{bmatrix}$$

Table 1 Optimal static output-feedback designs

Design case no.	Optimal output-feedback gain matrix with outputs $\{V, \alpha, q, \theta\}$		Closed-loop eigenvalues, rad/s	Optimum cost J_d
	K_w	K_d		
1[a]	[−0.94376, −0.76772, 1.0272, 1.8289]	Not applicable	$-2.3565 \pm 1.3017i$ $-0.3792 \pm 0.2459i$	4.4039
2[b]	[−0.94348, −0.81651, 1.0276, 1.8359]	[−0.89359, −0.77333, 0.97329, 1.7388]	$0.9310 \pm 0.0364i$ $0.9887 \pm 0.0073i$	4.40425
3[b]	[−1.5745, 0.0, 1.4355, 2.6308]	[−1.4593, 0.0, 1.3304, 2.4382]	$0.9075 \pm 0.0550i$ $0.9909 \pm 0.0086i$	4.60365
4[b]	[−0.75014, 0.0, −0.061008, 0.0]	[−0.75289, 0.0, −0.061231, 0.0]	$0.9796 \pm 0.0524i$ $0.9965 \pm 0.0164i$	18.2790
5[b]	[0.0, 0.0, 1.5642, 6.2791]	[0.0, 0.0, 1.4394, 5.7780]	$0.8983 \pm 0.1112i$ 0.99938, 0.98247	20.027
6[b]	[0.0, 0.0, 0.25775, 0.0]	[0.0, 0.0, 0.25412, 0.0]	$0.9588 \pm 0.0524i$ $0.9998 \pm 0.0016i$	3041.4

[a]Continuous-time design ($T = 0$ s). [b]Discrete-time design ($T = 0.03$ s).

Note that, even though the transfer function of the preceding system, $G(s)$ [or $G_z(z)$], is strictly proper (i.e., $D = D_d = 0$), the corresponding transfer function $G_w(w)$ in the w domain is proper (i.e., $D_w \neq 0$). Furthermore, the transfer function matrix $G_w(w)$ has a nonminimum-phase zero at 66.6667 that corresponds exactly to $w = 2/T$; whereas the same discrete-time transfer function $G(z)$ in the z domain does not have any transmission zeros. Hence, the w domain formulation exhibits directly the effect of nonminimum-phase zero due to the inherent delay introduced by a sample-and-hold device. Table 1 summarizes several discrete-time output-feedback designs that minimize the cost J_d in Eq. (17) to the specified initial state covariance X_o. Here, we use a feedback-law of the form $u(w) = K_w y_w(w)$ where the output vector $y_w(w)$ consists of the aircraft longitudinal state variables $\{V$(ft/s), α(deg), q(deg/s, θ(deg)$\}$. The optimal feedback-law K_d in discrete-time domain is subsequently reconstructed using the gain relation defined in the theorem. For design comparison, we also provide in design case 1 results of the continuous-time optimal state-feedback design. Notice that results of the discrete-time optimal state-feedback case (i.e., design case 2) are very much similar to the continuous-time design. These results are reasonable since the sampling time $T = 0.03$ s is small compared with the time-constant of the aircraft phugoid and short-period modes. The full-state-feedback design in design case 1 is determined from the solution of an algebraic Riccati equation. For the remaining design cases (2–6), optimal output-feedback gains are determined using the numerical algorithm implemented in Ref. 7. We eliminate the number of feedback output variables by constraining the feedback gains of the corresponding outputs to zero. The remaining set of feedback gains are then used to optimize the cost function J_d in Eq. (17). Clearly, the optimum cost J_d and the stability of the closed-loop system will gradually degrade as the number of feedback variables is reduced. Results of design case 3 indicate that one incurs little performance loss when feedback of angle of attack is omitted. This would be desirable since in practice accurate measurement of angle of attack is difficult and besides the measurement is fairly noisy. Design cases 4 and 5 show that either a combination of pitch rate and airspeed feedback, or a combination of pitch rate and pitch angle feedback would give nearly the same performance, with possibly a small improvement when airspeed feedback was used. Design case 6 represents a simple pitch damper design; the resulting performance is poor since the phygoid mode is nearly unobservable from the pitch rate measurement.

In the previous example, we have demonstrated the direct application of a constrained parameter optimization technique[7] originally developed for the continuous-time problem to solve an optimal discrete-time output-feedback control problem using the framework of the W-design methodology.

Conclusions

An alternative optimal control-law design method for sampled-data systems based on the w transform has been developed. The procedure has the potential of providing a unified framework for the synthesis of multivariable control systems in both the continuous and discrete-time domain. Hence, a single design tool can be used for both the continuous and discrete-time problems. An attractive feature of this formulation is that one can immediately use parameter optimization techniques originally developed for continuous-time systems to solve for optimum gains in an optimal discrete-time output-feedback control problem. We further anticipate that familiarity of designers with continuous-time domain design techniques would carry them in a straightforward manner to the synthesis of digital control-laws using the proposed W-design methodology. In this paper, an *exact* definition of a quadratic cost function in the w domain is given corresponding to a discrete-time optimal control problem. A set of necessary conditions for optimum static output-feedback designs have been derived leading directly to a solution method based on numerical gradient search.

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